Using Frequency-Matched Numerical Integration Methods for the Simulation of Dynamic Phasor Models in Power Systems

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***Abstract*—In this paper the use of frequency-matched linear numerical integration techniques for the simulation of systems modelled by the dynamic phasors approach (DPA) is proposed and investigated. Such methods show an increased accuracy and computational efficiency around the matched frequency. Such frequency matched methods are derived by analyzing the local truncation error in the frequency domain and determining the coefficients of the method in such a way so that the error is minimized around the matched frequencies. The aim in this way is to increase the accuracy and the efficiency of the numerical calculation, during the inherent fast oscillations featured in such systems modelled with dynamic phasors approach.**

***Index Terms*—Simulation, Dynamic Phasors Approach, Root-matching, Frequency-Matched Numerical Integration**

I. Introduction

For the combined simulation of the electromagnetic and electromechanical transients in power systems, various system variable representations are used. The electromagnetic transients programs often use the instantaneous value representation of system variables and system equations in the original three phases [1]. Another possibility is to use a variable representation on a rotating reference frame (DQ0), which leads to constant system quantities in balanced steady state. Another form of system variable representation is the so called dynamic phasors, which is based on the time varying Fourier Series approximation of the periodic or nearly periodic system quantities [2]. Dynamic phasors provide a suitable framework which has the capability to model power system behavior as accurate as EMTP simulations and at the same time, computationally more efficient. In particular, the dynamic phasors approach has been applied to model three-phase systems with electric machines, FACTS devices and power converters [3]–[5]. Mathematical details of the dynamic phasors approach will be discussed in Section II.

Regardless of the selected variable representation, in most of the cases, numerical integration techniques such as forward-Euler, backward-Euler, trapezoidal, Gear's method are employed. Generally, in three phase instantaneous value representation for modelling of power systems, the integration step size is very limited. Even if fast transients have decayed, the sinusoidal AC quantities of the electrical grid with the system frequency f_s cannot be integrated with the maximum possible integration step size $h_{max} = \frac{1}{2\,f_s}$, due to the lack of accuracy of the used numerical integration methods at the system frequency. The most commonly used integration methods, such as forward-Euler, backward-Euler, trapezoidal, Gear's method, are optimal for the numerical integration of low frequency signals, so that they are prone to some errors at higher frequencies if larger step sizes are used.

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One approach to overcome this problem is to use the so called *root-matching* method [6]. Numerical integration techniques generally discretize a continuous system $\mathbf{H}(s)$ by mapping it into an equivalent discrete time system $\mathbf{H}(z)$. The optimum discretization method should match the poles, zeros and final value of the difference equation to those of the actual continuous system. If these conditions are fulfilled, the difference equations are stable regardless of the step size, if the actual continuous system is also stable. This root-matching approach can be interpreted as an adjustment of the discretization (numerical integration) method to the eigenvalues of the system. This method has been successfully applied for the discretization of the RL, RC, etc. branch elements in EMTP instead of the traditionally employed trapezoidal discretization method [1].

In [7] the authors introduced another integration method adapted to the periodic steady state at system frequency f_s. This method allows the use of larger step sizes up to $h = \frac{1}{2\,f_s}$ for the numerical integration of three phase instantaneous electrical AC quantities.

Since we are using the dynamic phasor representation for power system modelling and simulation, our aim in this paper is to optimize the numerical integration method for variables represented by dynamic phasors.

As mentioned previously, in the root matching method, the poles, zeros and final value of the discrete time system are matched to those of the actual continuous system. The idea now is to relax this condition of root-matching and to match the numerical integration not exactly to the eigenvalues of the dynamic phasors but approximately to the oscillatory frequencies of the dynamic phasors. The aim in this way is to increase the accuracy and the efficiency of the numerical calculation during these inherent fast oscillations in systems modelled by dynamic phasors.

The paper will be organized as follows. First a general overview of the dynamic phasors approach will be given. This is followed by the investigation of the numerical integration techniques in the frequency domain, which will then allow us to derive the desired frequency-matched discretization methods. The accuracy and efficiency of these derived methods will be compared with the traditional approaches in a simple power systems example modelled with dynamic phasors.

II. Outlines of the Dynamic Phasor Approach

The main idea of the *Dynamic Phasors Approach* (DPA) is to approximate a possibly complex time domain waveform $x(\tau)$ in the interval $\tau \in (t-T, t]$ with a Fourier series representation of the form

$$x(\tau) \approx \sum_{k \in K} X_k(t)\, e^{j\,k\,\omega_s\,\tau} \tag{1}$$

$$X_k(t) = \frac{1}{T} \int_{t-T}^{t} x(\tau)\, e^{-j\,k\,\omega_s\,\tau} d\tau \tag{2}$$

where $\omega_s = 2\pi/T$ and $X_k(t)$ is the k^{th} time varying Fourier coefficient in complex form, also called *dynamic phasor*, and K is the set of selected Fourier coefficients which provide a good approximation of the instantaneous quantity. In this representation, since the interval under consideration slides as a function of time, the Fourier coefficients X_k are time varying. In this approach we are interested in cases where only a few coefficients provide a good approximation of the original waveform.

Some important properties of dynamic phasors are:

- The relation between the derivatives of $x(\tau)$ and the derivatives of $X_k(t)$, which is given in (3), where the time argument t has been omitted for clarity. This can easily be verified by differentiating the formula given in (1)

$$\left\langle \frac{dx}{dt} \right\rangle_k = \frac{dX_k}{dt} - j\,k\,\omega\,X_k \tag{3}$$

- The product of two time-domain variables equals a discrete time convolution of the two dynamic phasor sets of variables, which is given in (4).

$$\langle x\,y \rangle_k = \sum_{l=-\infty}^{\infty} (X_{k-l}\,Y_l) \tag{4}$$

For instance, let us assume a linear time invariant dynamic system modelled as a set of first-order differential equations

$$\dot{x}(t) = A\,x(t) + B\,u(t) \tag{5}$$

with an arbitrary sinusoidal forcing or source function $u(t)$ with frequency $k\,f_s$ and low frequency state $x(t)$. Assuming that the source function can be approximated by $u(t) \approx U_k(t)\,e^{j\,\omega_s\,t}$, the dynamic phasor model of the system becomes

$$\dot{X}_k(t) = (A - j\,k\,\omega_s\,I)\,X_k(t) + B\,U_k(t) \tag{6}$$

X_k and U_k are the new variables of the system and I is the identity matrix. The eigenvalues' oscillatory frequencies of the new system modelled with dynamic phasors are shifted by $-j\,k\,\omega_s$. This frequency shift gives rise to faster oscillations in the step response, which decay in stable systems, depending on the real part of the eigenvalues. These fast oscillations are inherent to dynamic phasor models. Our aim in the following sections is to derive a numerical integration method based on the trapezoidal method which is optimized for the numerical integration of such fast oscillations around system frequency ω_s or multiples of system frequency $k\,\omega_s$.

III. Trapezoidal Method

The Trapezoidal method is one of the most commonly used numerical integration techniques in the simulation of dynamic systems. It belongs to the family of linear multi-step methods and particularly to the Adam's Moulton methods (2^{nd} order Adam's Moulton method) [8]. A linear multi-step method is a method which approximates the true solution of the first order nonlinear ordinary differential equation of the form $\dot{x}(t) = f(x,t)$ at $x(t+h)$ by using a linear combination of previously computed values of the state variables x and their derivatives $\dot{x}$ and is given as

$$x(t+h) \approx \sum_{i=0}^{p} a_i\,x(t_n - i\,h) + \sum_{i=-1}^{p} b_i\,\dot{x}(t_n - i\,h) \tag{7}$$

In the case of 2^{nd} order Adam's Moulton method the time domain approximation of $x(t+h)$ is given by the following general expression

$$x(t+h) \approx a_0\,x(t) + b_{-1}\,\dot{x}(t+h) + b_0\,\dot{x}(t) \tag{8}$$

or in discrete form

$$x_{n+1} = a_0\,x_n + b_{-1}\,f(x_{n+1}) + b_0\,f(x_n) \tag{9}$$

with the coefficients $a_0 = 1$ and $b_0 = b_{-1} = \frac{h}{2}$.

Adam's Moulton Methods are implicit methods and therefore they have to be solved iteratively at every integration step. Iterative methods generally require an initial guess for the solution of nonlinear equations. A good initial guess reduces the number of required iterations for convergence which directly affects the computational efficiency of the method. Such an initial guess can e.g. be calculated by an explicit Adam's Bashforth method of the same order.

$$x_{n+1}^p = a_{0,p}\,x_n + b_{0,p}\,f(x_n) + b_{1,p}\,f(x_{n-1}) \tag{10}$$

with the coefficients $a_{0,p} = 1$, $b_{0,p} = \frac{3}{2}$ and $b_{1,p} = -\frac{1}{2}$. This process is often referred to as a predictor-corrector approach, since Adam's Bashforth method predicts the solution and Adam's Bashforth method corrects the solution. The predictor-corrector approach is commonly employed in conjunction with local truncation error estimation, which is then used for an adaptive step-size selection. If a predictor-corrector approach with same orders k is used the local truncation error can be approximated as $\varepsilon_l(t) = C\,|x^p - x|$, with x_p the predictor, x the corrector solution and C a constant dependent on the order of used method.

In the following, the focus will be on the error analysis in the frequency domain and numerical stability of the trapezoidal method as also performed in [7].

A. Error Analysis in Frequency Domain

A numerical integration method must satisfy certain criteria concerning numerical accuracy and numerical stability. Since a numerical integration approximates the true solution at each step, the error produced at each step can be defined as $|x(t_{n+1}) - x_{n+1}|$, under the assumption that the numerical solution x_n at t_n coincides with the real solution $x(t_n)$. This error introduced by advancing the solution from t_n to t_{n+1} in a single step is called also the *local truncation error* ε_l. This local introduced error is a measure of the numerical accuracy and should remain bounded during the simulation. Especially for the stationary solution, the local truncation error should become zero.

The generalized local truncation error of the 2^{nd} order Adam's Moulton method is given as

$$\varepsilon_l(t+h) = x(t+h) - [a_0 x(t) + b_{-1}\dot{x}(t+h) + b_0 \dot{x}(t)] \quad (11)$$

The Laplace transformation of the local truncation error yields

$$\Sigma_l(s) = X(s) \underbrace{\left(e^{s\,h} - a_0 - b_{-1} s e^{s\,h} - b_0 s\right)}_{E_l(s)} \quad (12)$$

Generally the numerical integration methods are optimized for DC stationary solutions meaning that the local truncation error becomes zero at $s = 0$ ($E_l(0) = 0$). This is also observable in Figure 1, which depicts the frequency response of the local error function $E_l(s)$ for the trapezoidal method. The derivation

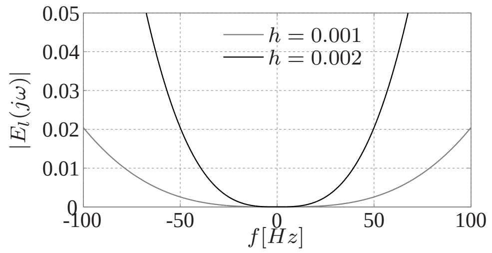


Fig. 1. Error frequency response $E_l(s)$ of Trapezoidal Method

of the trapezoidal method can also be accomplished in the frequency domain. The coefficients a_0, b_{-1} and b_0 in the general formulation of the 2^{nd} order Adam's Moulton can also be determined from the local error function in the frequency domain $E_l(s)$ in such a way, so that the local error function $E_l(s)$ has a triple root at DC ($s = 0$). This leads to the solution of the following three equations with three unknown coefficients a_0, b_{-1} and b_0

$$E_l(0) = 1 - a_0 = 0 \quad (13)$$

$$\frac{dE_l(0)}{ds} = h - b_{-1} - b_0 = 0 \quad (14)$$

$$\frac{d^2E_l(0)}{ds^2} = h^2 - 2\,b_{-1}\,h = 0 \quad (15)$$

The solution yields the well known coefficients of the trapezoidal method namely $a_0 = 1$, $b_{-1} = \frac{h}{2}$ and $b_0 = \frac{h}{2}$. Such an investigation of the local truncation error in the frequency domain opens new possibilities to derive new numerical integration methods which are not optimized at DC but at also other frequencies.

B. Numerical Stability Analysis

Numerical Stability is one of the most important properties of the numerical integration methods. Numerical stability ensures that the local truncation error remains bounded in subsequent integration steps. The standard method for testing numerical stability is to investigate how the eigenvalues of the continuous time system are mapped to the eigenvalues of the discrete time system. Numerical stable methods map the stable eigenvalues of the continuous time system with $Re\{s\} < 0$ to eigenvalues in discrete time system with $|z| < 1$. For the 2^{nd} order Adam's Moulton method, this can be calculated by replacing $z = e^{s\,h}$ in (12) and solving $E_l(s) = 0$ in z yielding

$$z = \frac{a_0 + b_0\,s}{1 - b_{-1}\,s}$$

Thus, the stability condition becomes

$$|z| = \left|\frac{a_0 + b_0\,s}{1 - b_{-1}\,s}\right| < 1 \quad (16)$$

In case of the trapezoidal method with pure real coefficients and $s = \sigma + j\,\omega$ the stability condition yields

$$\begin{aligned} (a_0 + b_0\,s)\,(a_0 + b_0\,s)^* &< (1 - b_{-1}\,s)\,(1 - b_{-1}\,s)^* \\ 1 + \frac{h^2}{4}\,|s|^2 + h\,Re\{s\} &< 1 + \frac{h^2}{4}\,|s|^2 - h\,Re\{s\} \\ Re\{s\} &< 0 \\ \sigma &< 0 \end{aligned}$$

This means, that stable continuous time systems ($Re\{s\} < 0$) are mapped directly to stable discrete time systems ($|z| < 1$) regardless of the selected step size h. Such numerical integration methods are absolute stable methods or A-stable methods.

After the local truncation error analysis in the frequency domain and the stability region analysis of the trapezoidal method, the focus in the following sections will be on the frequency matched methods and their error and stability region analysis.

IV. Frequency Matched Trapezoidal Method for Real Variables

If the trapezoidal method is used for the numerical integration of sinusoidal electrical AC quantities with system frequency (e.g. 50 Hz), they cannot be integrated with the maximum possible integration step size due to the lack of accuracy of the trapezoidal methods at the system frequency. As Figure 1 shows, that the local truncation of the trapezoidal method at 50 Hz is much higher than at DC frequencies, so that the step size must be kept small enough to achieve the same degree of accuracy at 50 Hz.

To overcome this problem, in [7] the authors propose the use of a new numerical integration method which is adapted to the periodic steady state of the electrical AC quantities. With this method, it is possible to compute the sinusoidal AC state variables with angular frequency ω_s at steady state exactly without introducing a local truncation error regardless of the selected step size h. This is achieved by investigating the local truncation error $\varepsilon_l(t)$ in the frequency domain. The coefficients a_0, b_{-1} and b_0 can be calculated in such a way that the roots of $E_l(s)$ in (12) are placed at frequencies other than DC. As stated in [7], the coefficients a_0, b_{-1} and b_0 can be calculated in such a way that the roots of $E_l(s)$ are placed at $-j\omega_s$, 0 and $j\omega_s$. The new coefficients a_0, b_{-1} and b_0 are determined by setting the following three equations to zero.

$$E_l(0) = 1 - a_0 \quad (17)$$

$$E_l(j\omega_s) = e^{j\omega_s h} - a_0 - jb_{-1}\omega_s e^{j\omega_s h} - jb_0\omega_s \quad (18)$$

$$E_l(-j\omega_s) = e^{-j\omega_s h} - a_0 + jb_{-1}\omega_s e^{-j\omega_s h} + jb_0\omega_s \quad (19)$$

The solution of (17-19) yields

$$a_0 = 1 \quad (20)$$

$$b_0 = \frac{j}{\omega_s}\frac{1-e^{j\,\omega_s\,h}}{1+e^{j\,\omega_s\,h}} = \frac{1}{\omega_s}\tan\left(\frac{\omega_s\,h}{2}\right) \quad (21)$$

$$b_{-1} = \frac{1}{\omega_s}\tan\left(\frac{\omega_s\,h}{2}\right) \quad (22)$$

With equation (9) and the new coefficients a_0, b_{-1} and b_0 the frequency matched trapezoidal method is fully defined. The corresponding predictor method for the frequency matched trapezoidal method is discussed in detail in [7]. In the following, a local truncation error analysis will be performed on the new method in the frequency domain.

A. Error Analysis in Frequency Domain

Figure 2 shows the frequency response of the local truncation error for the frequency matched trapezoidal method for real variables at $\omega_s = 2\,\pi\,50$ for different step sizes h. The desired frequency response of the local error function

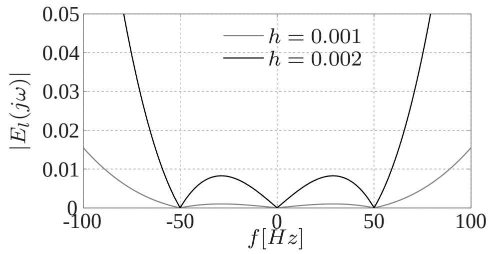


Fig. 2. Error frequency response $E_l(s)$ of Matched Trapezoidal for real variables

with three roots at $-j\omega_s$, 0 and $j\omega_s$ is observable. The local truncation error is zero at the matched frequencies regardless of the used step size, so that DC quantities and quantities with 50 Hz are computed without any error, which is the case in a power system at steady state without additional harmonics.

During electromechanical transients, however the frequency content of the state variables of the electromechanical system will not be exactly at the DC but in a narrow bandwidth around DC with a half-bandwidth of 2-3 Hz. Figure 3 shows the error frequency response of the trapezoidal method (denoted with the abbreviation **TR**) and of the frequency matched trapezoidal method for real signals (denoted with the abbreviation **TR-MR**). The frequency matched method shows a lack of accuracy around DC compared with the original method, hence the same degree of accuracy can only be achieved with smaller

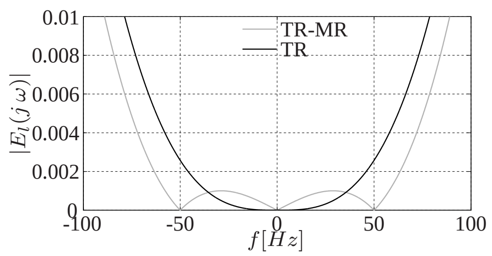


Fig. 3. Error frequency response $E_l(s)$ of trapezoidal and frequency matched trapezoidal method for real signals with $h = 0.001$ and $\omega_s = 2\,\pi\,50$

step sizes. However around the matched oscillatory frequency (50 Hz), the frequency matched method is much more accurate than the traditional trapezoidal method. As also reported in [7], in a system with a three-phase instantaneous value representation, the electrical AC quantities can be computed more efficiently with the frequency matched trapezoidal method than with the trapezoidal method.

After analyzing the accuracy of the frequency matched method and comparing it with the trapezoidal method, in the following, we will investigate the numerical stability of the matched method.

B. Numerical Stability Analysis

As discussed in section III-B, the general condition for numerical stability of the 2^{nd} order Adam's Moulton method is given in (16). In the case of the frequency matched trapezoidal method, the coefficients are also real valued. With (20-22), the stability condition becomes

$$\begin{aligned}
(a_0 + b_0\,s)\,(a_0 + b_0\,s)^* &< (1 - b_{-1}\,s)\,(1 - b_{-1}\,s)^* \\
a_0^2 + b_0^2\,|s|^2 + 2\,b_0\,Re\,\{s\} &< a_0^2 + b_{-1}\,|s|^2 - 2\,b_{-1}\,Re\,\{s\} \\
b_0\,Re\,\{s\} &< 0 \\
b_0\,\sigma &< 0
\end{aligned}$$

For stable systems ($\sigma < 0$) the condition for numerical stability becomes

$$\begin{aligned}
b_0 &> 0 \\
\frac{1}{\omega_s}\tan\left(\frac{\omega_s\,h}{2}\right) &> 0 \\
k\,\pi < \frac{\omega_s\,h}{2} &< (2\,k+1)\,\frac{\pi}{2} \\
\frac{k}{f_s} < h &< \frac{2\,k+1}{2\,f_s}
\end{aligned} \quad (23)$$

with $k \in \mathbb{N}$. The expression for $k = 0$, namely $h < \frac{1}{2\,f_s}$, reflects the Nyquist sampling theorem.

In contrast to the trapezoidal method, the stability range of the frequency matched trapezoidal method for real signals is limited. If the used step size is outside of the valid stability region (e.g. $\frac{1}{2\,f_s} < h < \frac{1}{f_s}$), the stable continuous system is mapped to an instable discrete time system. Thus the matched trapezoidal method for real signals is not an A-stable method.

The described frequency matched method has been successfully employed for the simulation of power system transients represented with three-phase instantaneous values, where the steady state is described by the periodic AC quantities [7]. The same method can also be applied to systems represented by dynamic phasors. In contrast to the three-phase instantaneous values, the dynamic phasor quantities are constant at steady state. Thus, the traditional trapezoidal method (and also other linear multi-step methods such as backward Euler or Gear's Method) can be used for the numerical integration of dynamic phasor quantities. The dynamic phasor representation is a baseband representation so that DC optimized (matched) numerical integration methods are accurate enough after fast transients have decayed. But as discussed previously, during fast transients with high frequencies, the trapezoidal method gets inaccurate and inefficient. The main idea of using such a frequency matched numerical integration is to increase the simulation accuracy and efficiency during the fast transients inherent to the dynamic phasors approach due to the shift of the eigenvalue's oscillatory frequency as mentioned in Section II.

The decreased accuracy of the frequency matched method around DC is not that critical, if it is used in conjunction with three-phase instantaneous values, as they are not DC at steady state but rather periodic with system frequency f_s. However, the use of the frequency matched method in conjunction with dynamic phasors requires a higher accuracy around oscillatory frequencies $k\,\omega_s$ and also a higher accuracy around DC.

In the following Section, we will derive another frequency matched method based again on the trapezoidal method, which is more accurate and numerically A-stable.

V. Frequency Matched Trapezoidal Methods for Complex Signals

In the previous Section, a method was derived, which is optimized for the numerical integration of real sinusoidal signals with frequency f_s or frequencies around f_s. However, real signals can also be represented by their analytical counterparts, namely complex or analytical signals, as they contain all the information about the original bandpass signal. The major advantage of using an analytical or complex signal is that the bandwidth of the original signal is reduced, as only positive (or negative) frequency components of the real signal are considered.

The frequency matched trapezoidal method described in the previous Section was derived for the numerical integration of periodic real bandpass signals. The derivation of the frequency matched trapezoidal method was performed by placing two of the three roots of the local error in frequency domain at $-j\,\omega_s$ and $j\,\omega_s$ as real valued sinusoidal signals' spectra with angular frequency ω_s are located at these frequencies. The representation of the real bandpass signals by their analytical counterparts brings one additional freedom in the placement of the roots, as the complex signal has only frequency component at positive or negative frequency. Hence, only one root has to be placed at the frequency $j\,\omega_s$ or $-j\,\omega_s$, the remaining two roots can be placed at DC.

In the following, the new method will be derived by using the same methodology as in previous sections, namely calculation of the coefficients a_0, b_{-1} and b_0 in such a way that the two roots of $E_l(s)$ in (12) are placed at DC and one at $j\,\omega_s$. The new coefficients a_0, b_{-1} and b_0 are determined by setting the following three equations to zero and solving in a_0, b_{-1} and b_0.

$$E_l(0) = 1 - a_0 \tag{24}$$

$$\frac{dE_l(0)}{ds} = h - b_{-1} - b_0 \tag{25}$$

$$E_l(j\,\omega_s) = e^{j\,\omega_s\,h} - a_0 - j\,\omega_s\,e^{j\,\omega_s\,h}\,b_{-1} - j\,\omega_s\,b_0 \tag{26}$$

The solution of (24-26) yields

$$a_0 = 1 \tag{27}$$

$$b_0 = \underbrace{\frac{h}{2}}_{b_{0,R}} + j\underbrace{\left(\frac{1}{\omega_s} - \frac{h}{2}\cot\left(\frac{\omega_s\,h}{2}\right)\right)}_{b_{0,I}} \tag{28}$$

$$b_{-1} = \frac{h}{2} - j\left(\frac{1}{\omega_s} - \frac{h}{2}\cot\left(\frac{\omega_s\,h}{2}\right)\right) \tag{29}$$

As a next step, an error analysis will be performed on the new frequency matched method and will be compared with to antecedently discussed methods.

A. Error Analysis in Frequency Domain

Figure 4 pictures the frequency response of the local truncation error for the frequency matched trapezoidal method for complex variables at $\omega_s = 2\,\pi\,50$ for different step sizes h. The desired frequency response of the local error

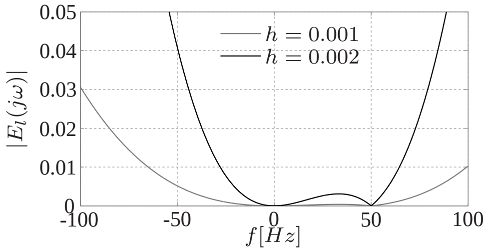


Fig. 4. Error frequency response $E_l(s)$ of Matched Trapezoidal for complex variables

function with three roots double root at DC and a single root $j\,\omega_s$ is observable. The local truncation error is zero at the matched frequencies regardless of the used step size, so that DC quantities and complex periodic quantities with 50 Hz are computed without any error. Figure 5 shows the error frequency response of the trapezoidal method, the frequency matched trapezoidal method for real signals and the frequency matched trapezoidal method for complex signals (denoted with the abbreviation **TR-MC**). As expected, placing two roots

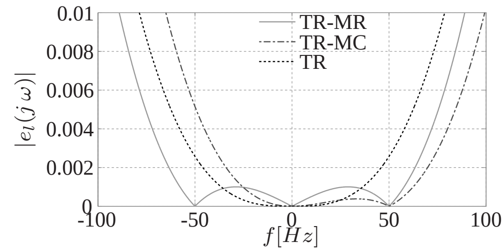


Fig. 5. Error frequency response $e_l(s)$ of trapezoidal, frequency matched trapezoidal methods for real and complex variables with $h = 0.001$ and $\omega_s = 2\,\pi\,50$

at DC in the error frequency response $E_l(s)$ increased the accuracy of the method for complex signals compared to the one for real signals. The frequency matched method for complex signals is more accurate than the frequency matched method for real signals around DC and also around $j\,\omega_s$.

Used in conjunction with predictor-corrector approach, the coefficients of the explicit predictor method can be determined in the same way by placing two roots at DC and one root at $j\,\omega_s$ yielding

$$a_{0,p}=1$$
$$b_{0,p}=\frac{h}{2}+\frac{\sin(\omega_s h)}{\omega_s}+j\left(\frac{h}{2}\left(\frac{\sin(\omega_s h)}{1-\cos(\omega_s h)}\right)-\frac{\cos(\omega_s h)}{\omega_s}\right)$$
$$b_{1,p}=\frac{h}{2}-\frac{\sin(\omega_s h)}{\omega_s}+j\left(\frac{\cos(\omega_s h)}{\omega_s}-\frac{h}{2}\left(\frac{\sin(\omega_s h)}{1-\cos(\omega_s h)}\right)\right)$$

B. Numerical Stability Analysis

Same procedures as in previous sections will be applied to define the region of stability of the new method. In the case of the frequency matched trapezoidal method, the coefficients b_0 and b_{-1} are complex valued but are also conjugate complex, meaning $b_{-1} = b_0^*$. With (27-29), the stability condition becomes

$$\begin{aligned}(a_0+b_0\,s)\,(a_0+b_0\,s)^* &< (1-b_{-1}\,s)\,(1-b_{-1}\,s)^*\\ a_0^2+|b_0\,s|^2+2\,Re\,\{b_0\,s\} &< a_0^2+|b_{-1}\,s|^2-2\,Re\,\{b_{-1}\,s\}\\ Re\,\{b_0\,s\} &< -Re\,\{b_0^*\,s\}\\ b_{0,R}\,\sigma-b_{0,I}\,\omega &< -(b_{0,R}\,\sigma+b_{0,I}\,\omega)\\ \sigma &< 0\end{aligned}$$

As in the case of trapezoidal method, this method is also an A-stable method mapping stable continuous time systems ($Re\,\{s\}<0$) directly to stable discrete time systems ($|z|<1$) irrespective of the selected step size h.

The described frequency matched method for complex signal is numerically stable and has also higher DC accuracy compared with the one for real signals. Thus, it can also be used in conjunction with the dynamic phasors.

VI. Test Cases

In the previous sections the focus was on the derivation of the frequency matched methods. The aim of this section is to compare the accuracy and efficiency of these methods with each other and verify the previously made conclusions by simulations. In this section, the derived numerical integration methods will be used for the simulation of two test systems. All three methods use a variable step-size selection algorithm based on the described local truncation error estimation with predictor-corrector approach. Simulations were obtained using Matlab 7.1 running on a Intel Pentium IV CPU with 3.80 GHz and 2 GB of RAM.

A. Example with Analytical Solution

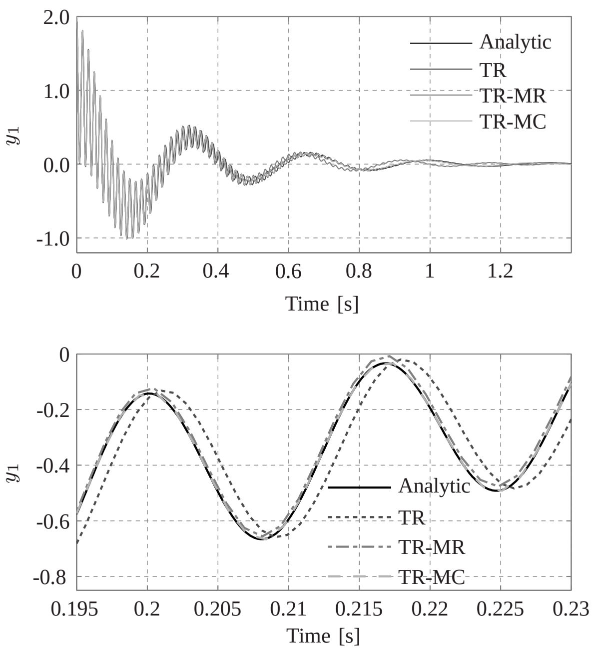
In this case study, the traditional trapezoidal method, frequency matched trapezoidal methods for real signals and for complex signal will be used for the numerical integration of a 4^{th} order ordinary differential equation system, whose analytic solution is also known. The differential equation system under consideration is given as

$$\begin{aligned}\dot{x}_1 &= -\sigma_1\,x_1-\omega_1\,x_2\\ \dot{x}_2 &= -\sigma_1\,x_2+\omega_1\,x_1\\ \dot{x}_3 &= -\sigma_2\,x_3-\omega_2\,x_4\\ \dot{x}_4 &= -\sigma_2\,x_4+\omega_2\,x_3\\ y_1 &= x_1+x_3\\ y_2 &= x_2+x_4\end{aligned}$$

with the parameter $\omega_1 = 2\,\pi\,3$, $\omega_2 = 2\,\pi\,60$, $\sigma_1 = 3$, $\sigma_2 = 6$ and the initial values $x_1(0) = 1$, $x_2(0) = 0$, $x_3(0) = 1$, $x_4(0) = 0$. The analytic solution of the initial value problem yields $x_1 = \cos(\omega_1\,t)\,e^{-\sigma_1\,t}$, $x_2 = \sin(\omega_1\,t)\,e^{-\sigma_1\,t}$, $x_3 = \cos(\omega_2\,t)\,e^{-\sigma_2\,t}$ and $x_4 = \sin(\omega_2\,t)\,e^{-\sigma_2\,t}$. The system is integrated using the TR, TR-MR (matched at $-j\,\omega_2$, 0 and $j\,\omega_2$) and TR-MC (matched at 0^2 and $-j\,\omega_2$). Figure 6 shows the results of the analytic solution and the numerical solution computed with three methods discussed in previous sections in the overall simulation interval $0 < t < 1.4$ and in a zoomed section with the required CPU simulation times. In the zoomed section we observe that the TR method has lack of accuracy during high frequency oscillations with $\omega_2 = 2\,\pi\,60$, where the frequency matched methods (TR-MR,TR-MC) give almost the accurate results during this fast transients. But if the fast transients have decayed and the system behavior is governed by low frequency transients with $\omega_1 = 2\,\pi\,3$, TR-MR method has inaccurate results. However, the TR-MC method computes the results with high degree of accuracy during fast and slow transients. Comparison of the overall CPU simulation times of all three integration methods shows that the TR-MC is also the fastest method for this simple test case.

B. SMIB system

The "Single Machine Infinite Bus" (SMIB) system [9] is used as a second test case. Figure 7 shows the single line diagram of the test system. In this test case, a single-phase to ground fault occurs at the BUS2 end of the LINE3 at 0.1 seconds and is removed after 0.20 seconds by disconnecting the line. The components of the system are modelled by dynamic phasors and are represented in the original three



TR	TR-MR	TR-MC
3.671 [s]	3.359 [s]	2.921 [s]

Fig. 6. Analytic solution of y_1 compared with numerical solution calculated with the three different integration methods and required CPU simulation times

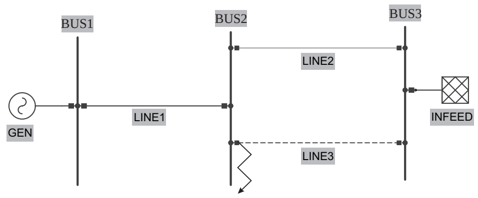
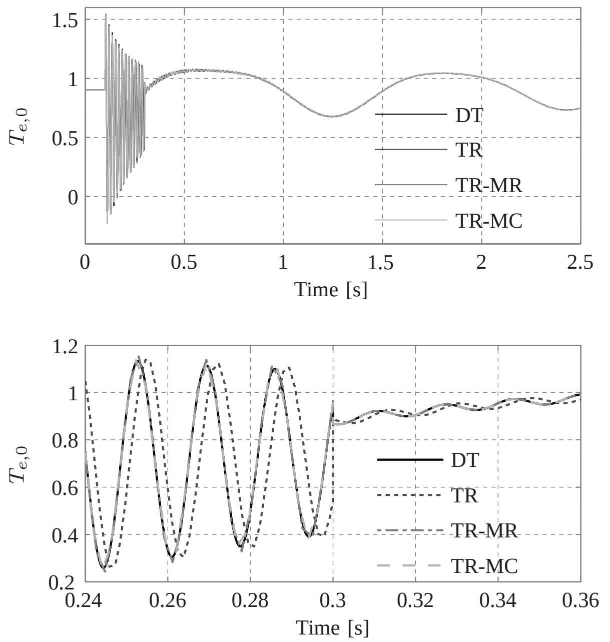


Fig. 7. Single line diagram of SMIB system

phases (ABC). The simulated system has overall 40 differential and 150 algebraic variables. Details of these models were reported in [10]. It is important to mention, that the approximating set of Fourier coefficients is selected as $K = \{1\}$ for the transmission line model and $K = \{0, 1, 2\}$ for the 9^{th} order synchronous machine model, as unbalanced conditions are of concern. The dynamic behavior of the test case is simulated by using the trapezoidal method and frequency matched trapezoidal methods for real and complex signals by using the same degree of accuracy for the three methods (error tolerance $\varepsilon_{tol} < 10^{-3}$) and with the same variable step-size algorithm based on the local truncation error estimation with predictor-corrector approach.

First simulation is done by using the traditional trapezoidal method (TR) for all dynamic phasors X_k. In the second simulation, the dynamic phasors with $k = 0$ are discretized using TR and with $k > 0$ using TR-MR with the frequencies matched to $-j\,k\,\omega_s$, 0 and $j\,k\,\omega_s$. In the third simulation, the dynamic phasors with $k = 0$ are discretized using TR and with $k > 0$ using TR-MC with the frequencies matched to $-j\,k\,\omega_s$ and 0^2 (with double root at 0).



DT	TR	TR-MR	TR-MC
186.125 [s]	25.410 [s]	49.750 [s]	26.045 [s]

Fig. 8. Evolution of the positive sequence electrical torque $T_{e,0}$ computed with TR, TR-MR and TR-MC (all with $\varepsilon_{tol} = 10^{-3}$) compared with the detailed simulation (TR with $\varepsilon_{tol} = 10^{-6}$) and required CPU simulation times

Figure 8 shows the evolution of the positive sequence electrical torque $T_{e,0}$ computed the TR, TR-MR and TR-MC compared with the detailed simulation (DT), which is computed with TR method by keeping the error tolerance quite low ($\varepsilon_{tol} < 10^{-6}$), which ensures an accurate simulation. The table in Figure 8 shows also the required CPU simulation times with different integration methods.

In the zoomed section in Figure 8 with fast transients around ω_s, the higher accuracy of the TR-MR and TR-MC methods compared to TR method is observable, where the TR-MR and TR-MC methods are in overall agreement with the detailed simulation results. The increased accuracy and efficiency of the TR-MC method at DC (also at ω_s) becomes noticeable in the comparison of the overall CPU simulation times. TR-MC method is two times faster than the TR-MR method due to its increased accuracy also for DC frequencies and as fast as the TR method in this test case.

VII. Conclusions

In this paper, a numerical integration method based on the trapezoidal method is derived for the simulation of systems modelled with dynamic phasors by minimizing the local truncation error around the oscillatory frequency ($k\omega_s$) of the dynamic phasors (X_k). The derived method shows a better performance regarding accuracy and efficiency during fast transients with the frequencies around $k\omega_s$ than the trapezoidal method. The use of complex/analytic signal representation facilitates the improvement of the numerical accuracy, efficiency and stability of the derived method compared with other frequency matched methods used for numerical integration of real signals.

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