Extracting Individual Thermostatic Load States from Aggregate Power Consumption Measurements via Moving Horizon State Estimation

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Abstract—We present an optimization-based state estimation method that allows us to estimate the states of individual thermostatically controlled loads (TCLs), such as space heaters, from measurements of the power consumption of small aggregations of TCLs. The state estimator can be used together with a controller to provide services to power systems such as frequency control. The main advantage of this method is that it is designed to work with existing communication infrastructure. We assume that aggregate power measurements are available from distribution substations every few seconds, while full state measurements are available from smart meters only every 20 minutes. We model TCLs as hybrid systems and propose a moving horizon state estimator (MHSE), which is formulated as a mixed-integer linear program. We demonstrate the performance of the MHSE in two case studies: (a) estimation of TCL states in the absence of external control actions, and (b) a power tracking problem with closed-loop control using broadcast control inputs. To demonstrate the robustness of the method, we conduct a parametric analysis with respect to aggregation size and diversity, process noise characteristics, and control trajectory characteristics. The results show that the method generally provides accurate estimates of TCL states, resulting in improved controller performance in most cases, and is implementable in real-time with reasonable computational power.

I. INTRODUCTION

Fluctuating renewable energy sources increase the need for power system services such as frequency control [1]. Traditionally, these services are provided by generators; however, there is a rising interest in using flexible electric loads to provide services [2] because they may be able to do so more effectively, at lower cost, and/or with less environmental impact. Active load participation in power systems is referred to as demand response (DR).

Thermostatically controlled loads (TCLs) such as air conditioners, space heaters, electric water heaters, and refrigerators operate with hysteresis controllers that include a temperature set-point and a dead-band. TCLs are good candidates for DR because (a) their power consumption can be shifted in time without compromising user comfort due to their thermal inertia; (b) they are easy to control; and (c) if aggregated, they can provide large amounts of DR. However, engaging large numbers of distributed TCLs in power system services may require significant investments in sensing, communication, and control infrastructure, in addition to new operational algorithms. Recent research has focused on methods to minimize the need for additional infrastructure by means of modeling, state estimation, and control strategies [3]–[6].

In this paper, we propose a novel moving horizon state estimation (MHSE) method that allows us to estimate the states of individual TCLs from measurements of the power consumption of small aggregations of TCLs. TCLs are modeled as stochastic hybrid systems (SHS), each with two states: a continuous state (temperature) and a discrete state (ON/OFF mode). Used together with a broadcast controller, the MHSE method allows an aggregator to control TCL populations to provide fast time scale services such as secondary frequency control (i.e. regulation) with high accuracy and minimal communication.

Several recent papers have investigated state estimation for TCL aggregations. Refs. [3], [4] proposed a method of modeling aggregations of TCLs with stochastic linear models based on Markov chains. One of the benefits of these models is that they can be used together with Kalman Filters for state estimation. Ref. [6] proposed a state estimation approach based on a similar model and an MHSE method, while [5] proposed a four state aggregate system model, similar to that in [7], and a particle filter to estimate aggregate TCL states. However, with each of these approaches one is only able to estimate the fraction of TCLs in discrete bins in the state space, not the states of individual TCLs. Additionally, [6] did not consider measurement noise.

Estimating the states of individual TCLs instead of the states of aggregate models could improve control performance. Individual state estimates help us better estimate the effect of the TCLs’ internal controllers and the aggregator’s external control actions on the TCL aggregate power consumption. Separate state estimation and control is not optimal for SHS since the separation principle does not apply to such systems. However, it is common practice to heuristically separate the two tasks in the interest of simplicity. Control performance is generally improved with better state estimates; however, this must be verified through testing. SHS state estimation problems are often solved with ‘multiple model’ estimation schemes that involve a filter for each mode [8]. However, with large numbers of modes, as in our problem, these approaches are infeasible [9].

Our contributions are threefold. First, since the SHS TCL aggregation model is not amenable to common state estimation methods, we derive a stochastic mixed logical dynamical (MLD) model [10] from the SHS model. This allows us to represent a TCL aggregation as a linear system with mixed-integer linear inequalities, which can be used
within a mixed-integer linear program. To do this, we extend the MLD model for a single deterministic TCL proposed in [11] by a) considering external forcing from time-varying ambient temperature and process noise, b) expanding the model to handle TCL aggregations, and c) incorporating the effect of external broadcast control actions. Second, we propose an MHSE method based on mixed-integer linear programming (MILP) that is able to estimate the states of individual TCLs from aggregate power measurements.

Ref. [12] shows how MHSE can be used for MLD systems, while [13] derived sufficient conditions for asymptotic convergence of MHSE algorithms. However, the conditions are proved only for a deterministic system, and so we do not apply those results here. Instead, we investigate the effect of parameters that influence convergence – estimation horizon length, state penalties, etc. – empirically. Third, we illustrate how the MHSE method improves controller performance by conducting a parametric analysis with respect to aggregation size and diversity, process noise characteristics, and control trajectory characteristics.

The remainder of the paper is organized as follows. Section II describes the state estimation problem and Section III introduces the SHS and MLD models. In Section IV we describe the control scheme and in Section V we detail the MHSE method. Section VI presents two case studies and in Section VII, we give concluding remarks.

II. PROBLEM DESCRIPTION

We consider an aggregator managing a population of TCLs in a distribution network. We assume that aggregate power measurements from distribution substations are available to the aggregator at each time step. We also assume two-way communication between the aggregator and each TCL, for example, via smart meters, with latency and bandwidth constraints. Due to these constraints, TCL-level state measurements are received less frequently than aggregate power measurements (e.g., every 20 minutes), and control signals can only be broadcasted to the TCLs. The assumed measurements and their frequency are shown in Figure 1.

Our goal is to estimate the individual TCL states between two consecutive TCL-level state measurements using load models and aggregate power measurements in order to improve control performance. This is a challenging task due to process and measurement noise. Process noise includes plant-model mismatch and errors in predictions of external forcing such as ambient temperature and consumer behavior. Measurement noise includes errors in aggregate power measurements, which can be computed from substation power measurements by subtracting the predicted non-controllable load connected to the same substation.

III. MODELING

A. Individual TCL Modeling

Processes that evolve according to continuous dynamics, discrete dynamics, and logic rules can be modeled as hybrid systems [11]. Here, we use the two-state hybrid TCL model developed in [14]–[16]. Denote the TCL temperature at time step by \( x_{c,t} \in \mathbb{R} \) and the ON/OFF state at time step \( t \) by \( x_{l,t} \in \{0, 1\} \). A heating TCL’s stochastic discrete-time dynamics can be expressed as follows:

\[
\begin{align*}
  x_{c,t+1} &= a x_{c,t} + b u_{t} x_{c,t} + f T_{a,t} + w_{t}, \quad (1) \\
  x_{l,t+1} &= \begin{cases} 
  0 & \text{if } x_{c,t+1} \geq M \\
  1 & \text{if } x_{c,t+1} \leq m \\
  x_{l,t} & \text{otherwise} 
  \end{cases} \quad (2)
\end{align*}
\]

where \( a = e^{-\Delta t/(CR)} \), \( b = (1-a)RC \), \( f = (1-a) \), \( \Delta t \) is the discretization time step, \( C \) is the thermal capacitance, \( R \) is the thermal resistance, \( C_p \) is the Coefficient of Performance (COP), \( P_n \) is the rated power, \( u_{t} \in [0, 1] \) is the fraction of the rated power consumed by the TCL at time step \( t \) if it is ON, \( T_{a,t} \) is the ambient temperature, and \( w_{t} \) is the process noise. Additionally, \( M = T_{sp} + 0.5T_{db} \) and \( m = T_{sp} - 0.5T_{db} \) are the upper and lower temperature dead-band limits, respectively, where \( T_{sp} \) is the thermostat temperature set-point and \( T_{db} \) is the dead-band width.

The SHS in (1)-(2) can be described using the MLD framework. Following the approach proposed in [11], we introduce the auxiliary binary variables \( \delta_{1,t}, \delta_{2,t}, \delta_{3,t}, \delta_{4,t} \) defined as follows:

\[
\begin{align*}
  [\delta_{1,t} = 1] &\leftrightarrow [x_{c,t} \geq M], \quad (3) \\
  [\delta_{2,t} = 1] &\leftrightarrow [x_{c,t} \leq m], \quad (4) \\
  \delta_{3,t} &= x_{l,t+1}, \quad (5) \\
  \delta_{4,t} &= x_{l,t}. \quad (6)
\end{align*}
\]

Additionally, since (1) is bilinear between \( x_{l,t} \) and \( u_{t} \), we introduce the auxiliary continuous variable \( z_{t} = x_{l,t} u_{t} = \delta_{4,t} u_{t} \) to achieve linearity of the model. Then, by defining \( x_{t} := [x_{c,t} x_{l,t}]^{T} \) and \( \delta_{t} := [\delta_{1,t} \delta_{2,t} \delta_{3,t} \delta_{4,t}]^{T} \), (1) can be rewritten as:

\[
\begin{align*}
  x_{t+1} &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} x_{t} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \delta_{t} + \\
  &+ \begin{pmatrix} b & 0 \\ 0 & f \end{pmatrix} z_{t} + \begin{pmatrix} 0 & 0 \end{pmatrix} T_{a,t} + \begin{pmatrix} w_{t} \\ 0 \end{pmatrix}, \quad (7)
\end{align*}
\]
Note that (7) is similar to (11a) in [11], except that we include two additional terms: ambient temperature and process noise.

The TCL internal hysteresis controller (2) can be described by the following logic relations:

\[ [\delta_{1,t} = 1] \rightarrow [\delta_{3,t} = 0] , \]  
\[ [\delta_{2,t} = 1] \rightarrow [\delta_{3,t} = 1] , \]  
\[ [\delta_{1,t} = 0] \land [\delta_{2,t} = 0] \rightarrow [\delta_{3,t} = \delta_{4,t}] , \]  

which can be transformed into mixed-integer linear inequalities:

\[ E_2 \delta_t + E_3 z_t \leq E_1 u_t + E_4 x_t + E_5 . \]  

The definitions of \( E_1, E_2, E_3, E_4, \) and \( E_5 \) can be found in the Appendix. Note that to keep the formulations generic, we have modeled the TCL power consumption \( u_t \) as a continuous variable. However, in this paper we consider the most common case where TCLs operate at rated power when they are ON, i.e., \( u_t = 1 \). With this simplification, the auxiliary variable \( z_t \) is not needed.

### B. TCL Aggregation Modeling

A heterogeneous aggregation of \( n_{agp} \) TCLs can be modeled by drawing the TCL parameters, i.e., \( R, C, C_p, P_n, M \) and \( m \), from suitable distributions and stacking together models of individual TCLs, leading to the following state-space representation:

\[ x_{t+1} = Ax_t + B_2 \delta_t + B_3 z_t + F_t + w_t , \]  
\[ y_t = \begin{cases} C_1 x_t , & \text{if } t = j T_m , j \in \mathbb{N} \\ C_2 x_t + v_t , & \text{otherwise} \end{cases} , \]  

\[ E_2 \delta_t + E_3 z_t \leq E_1 u_t + E_4 x_t + E_5 , \]  

where \( x_t, \delta, z_t, w_t, E_5 \) are stacked vectors, e.g., \( x_t = [x_1^T, \ldots, x_n^T]^T; A, B_2, B_3, F_t, E_1, E_2, E_3, E_4 \) are block diagonal matrices with the matrices of the individual TCLs on the diagonals; \( C_1 = I; C_2 = [0, P_n^a, 0, P_n^a, \ldots, 0, P_n^{a_w}]; u_1 = 1; v_t \) is measurement noise; and \( T_m \) is the period of TCL-level measurements. Note that the output of the system depends on the time step; every \( T_m \) steps full state information is available, but at every other time step only the aggregate power is measured. Also, note that in (13) we consider noise in aggregate power measurements, whereas TCL-level state measurements are assumed noise-free.

### IV. CONTROL PROBLEM

We use the closed-loop rule-based control algorithm proposed in [17] to make the TCL aggregation track a power trajectory. At each time step \( t \), the controller receives a measurement of the aggregate power consumption of the aggregation \( P_{agg,t} \) and, based on the desired set-point \( P_{set,t} \), calculates the required change in power

\[ \Delta P_t = P_{set,t} - P_{agg,t} . \]  

However, at each time step, the actions of the internal thermostats will result in a change in aggregate power consumption \( \Delta P_{int,t} \). Therefore, the controller must estimate \( \Delta P_{int,t} \) so that it can calculate the effective change in power required by external control actions

\[ \Delta P_{eff,t} = \Delta P_t - \Delta \hat{P}_{int,t} , \]  

where \( \Delta \hat{P}_{int,t} \) can be estimated from state estimates \( \hat{x}_t \) calculated with the MHSE algorithm described in Section V. If \( \Delta P_{eff,t} < 0 \) additional OFF switching is required, whereas if \( \Delta P_{eff,t} > 0 \) additional ON switching is required. The TCLs that will be switched are determined according to a priority list based on their estimated State of Charge (SOC)

\[ SOC_t = \frac{x_{c,t} - m}{M - m} , \]  

where \( x_{c,t} \) is the TCL's estimated temperature.

At each time step, the controller broadcasts a pair \( [SOC_{th,t}, \delta_t] \), where \( SOC_{th,t} \in [0,1] \) is the \( SOC_t \) of the last TCL that enters the priority list, and \( \delta_t \in \{0,1\} \) is a signal indicating whether an increase in consumption \( (\delta_t = 1) \) if \( \Delta P_{eff,t} > 0 \) or a decrease in consumption \( (\delta_t = 0) \) if \( \Delta P_{eff,t} < 0 \) is required. The TCLs that are outside of their dead-band are not controllable and ignore the control signal, whereas the rest respond based on their SOC. For each TCL, \( SOC_{th,t} \) can be mapped to a temperature threshold

\[ x_{th,t} = SOC_{th,t} (M - m) + m . \]  

The desired control actions can be described by the state transitions in Tables Ia and Ib, which can be incorporated into the MLD framework of Section III by introducing \( [\delta_{l,t} = 1] \leftrightarrow [x_{c,t} \leq x_{th,t}] \). However, an equivalent formulation can be obtained without adding a new auxiliary variable in the following way. Set \( \bar{M} = x_{th,t} \) and \( \bar{m} = m \) if \( s_t = 0 \), and \( \bar{M} = M \) and \( \bar{m} = x_{th,t} \) if \( s_t = 1 \). With this notation and given that \( SOC_{th,t} \in [0,1] \), Tables Ia and Ib are equivalent to Table Ic by inspection.

Now, the state transitions of Table Ic can be described by the following logic relations:

\[ [\delta_{1,t} = 1] \leftrightarrow [x_{c,t} \geq \bar{M}] , \]  
\[ [\delta_{2,t} = 1] \leftrightarrow [x_{c,t} \leq \bar{m}] , \]
along with (5), (6), (8)–(10). Therefore, the external control actions can be directly incorporated into the MLD framework of Section III using (11). The only difference is that matrices $E_2$ and $E_5$, which now depend on $M$ and $\hat{m}$, are time-varying. The external control can be seen as a dynamic tightening of a TCL’s dead-band, which is visualized in Fig. 2. This control approach is similar to approaches based on temperature set-point control, e.g., as in [18]; however, in our approach user comfort is always guaranteed since $\hat{M} \leq M$ and $\hat{m} \geq m$. The control loop including the state estimation procedure is shown in Fig. 3.

V. STATE ESTIMATION PROBLEM

We propose an MHSE algorithm to estimate TCL states when TCL-level measurements are unavailable. At each time step $t \neq jT_m$, $j \in \mathbb{N}$ we solve a multi-period mixed-integer optimization problem with TCL temperatures, ON/OFF states, auxiliary binary variables, process noise, and measurement noise as optimization variables. Define the optimization vector at time step $t$ as $\mathbf{x}_{t}^{op} := [x_{t-N+1}^t, \tilde{x}_{t-N+1}^t, \ldots, \tilde{x}_{t}^t]$, where $\tilde{x}_{t}^t := [\tilde{x}_{k|t}, \tilde{z}_{k|t}, \tilde{w}_{k|t}, \tilde{v}_{k|t}], k \in [t-N+1,t]$. $N$ is the estimation horizon, and $\tilde{x}_{k|t}$ denotes the estimate of $x_k$ at time step $k$ using measurements up to time step $t$. Note that $\tilde{x}_{t-N+2|t}, \ldots, \tilde{x}_{t}^t$ can be determined from $\tilde{x}_{k|t}, k \in [t-N+1,t]$, and therefore these additional optimization variables are not needed. With this notation, the optimization problem can be expressed as follows:

$$\min_{\mathbf{x}_{t}^{op}} \sum_{k=t-N+1}^{t} m_1 \left| \hat{y}_{k|t} - y_k \right| + m_2 \left| \hat{w}_{k|t} \right|_1$$
$$+ m_3 \left| \hat{v}_{k|t} \right| + \sum_{k=t-N+1}^{t-1} m_4 \left| \hat{x}_{k|t} - \hat{x}_{k|t-1} \right|_1$$
$$+ m_5 \sum_{i=1}^{n_w} \left| \hat{W}_{k|t}^i - W^i \right| + m_6 \left| \hat{V}_{k|t} - V \right|,$$  

(21)

s.t.  
$$\hat{\mathbf{x}}_{k+1|t} = A \hat{\mathbf{x}}_{k|t} + B_2 \hat{\mathbf{d}}_{k|t} + B_3 \hat{\mathbf{z}}_{k|t} + F_4 + \hat{\mathbf{w}}_{k|t}, \quad (22)$$
$$\hat{y}_{k|t} = C_2 \hat{\mathbf{x}}_{k|t} + \hat{\mathbf{v}}_{k|t}, \quad (23)$$
$$E_2 \hat{\mathbf{d}}_{k|t} + E_3 \hat{\mathbf{z}}_{k|t} \leq E_4 \mathbf{u}_t + E_5 \hat{\mathbf{x}}_{k|t} + E_6,$$  

(24)
$$\hat{\mathbf{x}}_{k|t} = x_k, \quad \forall k \in [jT_m, jT_m + N - 1], \quad (25)$$

where $\hat{\mathbf{d}}_{k|t} \in \{0,1\}^{4n_w}$, $\hat{\mathbf{z}}_{k|t} \in \mathbb{R}^{n_w}$, $\hat{\mathbf{w}}_{k|t} \in \mathbb{R}^{n_w}$, $\hat{\mathbf{v}}_{k|t} \in \mathbb{R}$, $m_1$ to $m_6$ are weighting factors, $W^i$ is a known statistic on the process noise of TCL $i$, $V$ is a known statistic on the measurement noise, and $\hat{W}_{k|t}^i$ and $\hat{V}_{k|t}$ are the current estimates of those statistics computed from $\hat{\mathbf{w}}_{k|t}$ and $\hat{\mathbf{v}}_{k|t}$.

The first term of (21) minimizes the difference between the output calculated from the estimated states (23) and the measured aggregate power. The second and third terms penalize the process and measurement noise, where $m_2$ and $m_3$ may be a function of the noise variance/covariance matrices, $Q$ and $R$, if they are known. Note that there is no assumption on the probability distributions of the noise. The fourth term is needed to link the current estimation problem to the results of the previous estimation problems; the choice of $m_4$ is critical for the convergence of the estimator [13]. The fifth and sixth terms require that the current noise statistics are equivalent to their known values. However, if no noise statistics are known, the method can be applied without the last two terms.

For $k \in [jT_m, jT_m + N - 1]$, $j \in \mathbb{N}$, noise-free TCL-level measurements are available, which are taken into account by introducing the equality constraint (25), and setting $m_4 = 0$ in (21). For $t \geq jT_m + N$, (25) is not considered and $m_4 \neq 0$.

Let $n_q = 4$ denote the number of auxiliary binary variables per TCL, $n_w = 1$ denote the number of process noise variables per TCL, and $n_v = 1$ denote the number of measurement noise variables per time step. Then, the total number of variables is $n_{var} = 2n_q + n_w(n_q + n_w) + n_v$, of which $n_{bi} = n_q + n_q n_q$ are binary variables. Therefore, the MHSE problem is a large scale mixed-integer optimization problem, even for small TCL aggregations and estimation horizons. For example, with $n_q = 20$ and $N = 10$, the resulting problem has 820 binary variables. Due to the size of the problem, we use a 1-norm minimization in (21), which can be reformulated as a MILP instead of least squares minimization as in [11]–[13], which leads to a quadratic integer program.

VI. CASE STUDIES

We demonstrate the performance of the MHSE method via two case studies. In case study A, we estimate the states of a TCL aggregation in the absence of external control actions to gain insight into the estimation mechanism. In this case, the TCLs are controlled only by their internal thermostats. In case study B, we perform state estimation on a TCL aggregation that is externally controlled to track a power trajectory. The purpose of this investigation is to quantify controller performance improvement with better.

![Fig. 2. Dynamic tightening of TCL dead-band under control actions, where green denotes the active dead-band for each case.](image)

![Fig. 3. Control loop including the moving horizon state estimator.](image)
TCL state estimates. To understand how process noise affects
the estimation quality, we set the measurement noise to zero
in both case studies. We also disregard the last two terms of
(21) by setting $m_5 = m_6 = 0$. We use space heating TCLs
and parameterize them by drawing the parameters from the
uniform probability distributions listed in Table II.

The optimal estimation period depends upon the aggre-
gation size and, in practical applications, would likely be
determined by real-time computational limitations. For these
reasons, we do not choose $N$ based on the observability
tests proposed in [11], but select it empirically. However,
we investigate how the choice of $N$ affects the quality of
the estimation by conducting a parametric analysis in case
study A. In addition, case study B was repeated for different
aggregations, noise characteristics, and control trajectories
to assess the robustness of the method. The weighting
factors $m_1$ to $m_6$ in (21) are all chosen equal to 1 since
preliminary simulations showed that this choice leads to good
performance. All simulations were done in MATLAB using a
4 core machine (2.83 GHz) with 8 GB RAM, and the MHSE
problem was solved using CPLEX.

A. Results for Case Study A

To understand the details of the MHSE method, we first
consider an aggregation of 20 TCLs and run hourly simu-
lations with a time step of 10 seconds. Only integer values
of $P_0$ are used, which complicates the estimation process
since different TCLs may appear similar to the estimator. We
assume that the process noise for each TCL follows a normal
distribution with mean value $\mu = 0$ and standard deviation
$\sigma = 10^{-3}$, and that the TCL states at the beginning of the
simulation are known, based on the discussion of Section II.
The estimation horizon is fixed to $N = 10$.

We compare the MHSE against a simple model-based pre-
diction, which assumes zero process noise. Fig. 4 shows the
actual, estimated, and predicted temperature trajectories of a
single TCL. During the first 36 minutes, the MHSE behaves
only marginally better than the model-based predictor. At this
point, the estimator notices a significant mismatch between
the expected and actual aggregate power. Afterwards, the
temperature estimate converges to the actual state; however, the
MHSE temperature estimation error is kept at lower, though
correctly estimated by MHSE, whereas the predictor pro-
duces significant errors. Around 42 minutes, the ON/OFF
state estimates diverge from the actual state; however, later on
the estimation error goes to zero. Therefore, even in case
of temporary wrong estimates, the ON/OFF state estimates
evolve to the actual states. The temperature prediction error generally increases with time, whereas the
MHSE temperature estimation error is kept at lower, though
non-zero, values. A perfect temperature estimation is not
possible because the estimator retrieves information only
when differences between predicted and measured power
occur. In between, the temperature estimation error might
grow depending on the process noise.

We next investigate the applicability of the MHSE method
for larger aggregations and its performance with different
estimation horizons. Fig. 6 shows results from simulations
with $n_{ap} = \{10, 20, 30\}$ and $N = \{5, 10, 15, 20\}$. To ensure
a fair comparison, exactly the same process noise data were
used to simulate a given aggregation for each different $N$. In
all cases, an horizon of $N = 10$ performs very well, while
further increasing $N$ does not reduce the estimation errors.
Shorter horizons degrade estimation quality for aggregations
of 20 or 30 TCLs; however, for very small aggregations of 10
TCLs even an horizon of $N = 5$ is enough. Note that since
we have considered specific TCL aggregations and process

\begin{table}[h]
\centering
\caption{Space heater parameters, adapted from [3], [4]}
\begin{tabular}{|c|c|c|c|}
\hline
$P_0$ & $C_p$ & $T_{ap}$ & $T_{dB}$ \\
\hline
\hline
\end{tabular}
\end{table}

\begin{align}
\text{MAE}_t &= \frac{1}{n_{ap}} \sum_{i=1}^{n_{ap}} |\hat{x}_{c,t}^{i} - \tilde{x}_{c,t}^{i}|. 
\end{align}

The evolution of estimation errors over time is shown in
Fig. 5. The ON/OFF state of each TCL is almost always
correctly estimated by MHSE, whereas the predictor pro-
duces significant errors. Around 42 minutes, the ON/OFF
state estimates diverge from the actual state; however, later on
the estimation error goes to zero. Therefore, even in case
of temporary wrong estimates, the ON/OFF state estimates
eventually converge to the actual states. The temperature
prediction error generally increases with time, whereas the
MHSE temperature estimation error is kept at lower, though
non-zero, values. A perfect temperature estimation is not
possible because the estimator retrieves information only
when differences between predicted and measured power
occur. In between, the temperature estimation error might
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A perfect temperature estimation is not possible because the estimator retrieves information only when differences between predicted and measured power occur. In between, the temperature estimation error might grow depending on the process noise.
noise time series, these results are only meant to provide
intuition. More general conclusions regarding the best choice
of \( N \) could be obtained by Monte Carlo simulations.

The problem (21)-(25) can be solved quickly for the
aggregation sizes considered here. Table 6 shows how the
average computation time (first number) and the maximum
computation time (second number) depend on \( n_{ap} \) and \( N \).
Not surprisingly, increasing \( n_{ap} \) or \( N \) leads to longer
solution times. Note that increasing \( N \) has a larger impact on
computation time since this introduces more binary variables
in the problem. The reported computation times are likely
acceptable for real-time estimation applications.

### B. Results for Case Study B

Our ultimate question is how much the controller perfor-
ance improves with better TCL state estimates. For this
purpose, we use the control scheme of Section IV to track
\( P_{set} \) considering two cases: (a) the model-based prediction
approach and (b) the MHSE method. The power trajectory
\( P_{set} \) is calculated by superimposing a random signal upon the
predicted TCL aggregation baseline, i.e., the power trajectory
without external control actions, as in [17], [19]. We run
simulations with horizon of 20 minutes and use exactly the
same 20 TCL aggregation and process noise data as in the
previous subsection.

For ease of interpretation, the absolute tracking errors for
the prediction and MHSE were smoothed using a simple
moving average with a window size of 1 minute and are
shown in Fig. 7a. The prediction leads to a MAE of 6.40 kW,
whereas MHSE to a MAE of 4.04 kW, which represents
an improvement of about 37%. Fig. 7b shows the broad-
casted control signal \( SOC_{th} \) for both cases. One can see
that improved state estimates with MHSE also lead to less
aggressive control signals, as compared to the model-based
prediction approach.

Figure 7c shows the evolution of estimation errors over
time. The ON/OFF state estimation error of MHSE is gener-
ally larger than in case study A, which is due to the external
control actions. Also, note that there exist intervals when the
MHSE ON/OFF state and temperature errors are larger than
the prediction errors. However, the controller performs better
with MHSE than with prediction during the same intervals.
This indicates that even when the MHSE provides poor TCL-
level state estimates, it still improves controller performance.

To provide evidence that the MHSE generally improves
controller performance, we repeat the analysis for different
combinations of TCL aggregations, process noise charac-
teristics, and control trajectories. We consider normally dis-
tributed zero mean process noise with three different standard
deviations \( \sigma = \{ 5 \cdot 10^{-4}, 10^{-3}, 5 \cdot 10^{-3} \} \), which we refer to as “low”, “medium”, and “high” noise. We also consider
two control trajectories: a) a “moderate trajectory” based
on a Swiss secondary frequency control signal and b) an
“aggressive trajectory” in which we superimpose zero mean
random noise drawn from a normal distribution with standard
deviation \( \sigma_f = 10 \text{kW} \) on top of the secondary frequency
control signal. For each of the six combinations we run 200
simulations with randomly sampled TCL aggregations of size
\( n_{ap} = 20 \).

Figure 8 shows the resulting improvements in controller
of individual loads in small aggregations of thermostatically controlled loads (TCLs). The method assumes a two-way constrained communication infrastructure between an aggregator and each TCL, where control signals can only be broadcasted and TCL-level measurements can be received at a lower frequency than aggregate power measurements. The proposed method handles both process and measurement noise; however, only process noise was explicitly investigated in the case studies. We conducted simulations for a case without external control and a power trajectory tracking application. Our results show that state estimates are generally accurate and result in controller performance improvement in most cases. Additionally, our investigations revealed a number of interesting observations. First, the MHSE method works better with diverse TCL aggregations and aggressive control trajectories. Second, good tracking can be achieved even with poor state estimates for individual TCLs. The reason is that MHSE cannot easily distinguish among TCLs with similar parameters, in particular if their power ratings are identical. Clustering of TCLs based on their parameters could be used to modify this behavior. Third, for aggregation sizes up to 30 TCLs, an estimation horizon $N = 10$ seems to be a reasonable tradeoff between estimation accuracy and computation time.

There are many avenues for future work. Preliminary simulations show that measurement noise can be in principle handled by the proposed method, but this would probably require longer $N$ and lead to longer solution times. Also, penalizing process and measurement noise using $Q$ and $R$ in (21) only, might not be enough for good performance. A potential solution would be to filter the aggregate power measurements before inserting them into the MHSE problem.

**APPENDIX**

As in [11], the matrices $E_1, E_2, E_3, E_4,$ and $E_5$ in (11) are defined as follows:

$$E_1 = [0 0 0 0 0 0 0 0 0 0 0 0 0 1 -1]^T, \quad (27)$$
$$E_2 = \begin{pmatrix} 0 & M_x - m & 0 & 0 \\ 0 & m_x - m - \epsilon & 0 & 0 \\ M - M_x & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -M_u \\ 0 & 0 & m_u & 0 \\ 0 & 0 & 0 & m_u \\ 0 & 0 & 0 & -m_u \\ 0 & 0 & 0 & M_u \end{pmatrix}, \quad (28)$$

$$E_3 = [0 0 0 0 0 0 0 0 0 0 1 -1 -1]^T, \quad (29)$$

**VII. CONCLUDING REMARKS**

This paper presented a moving horizon state estimation (MHSE) method to extract temperatures and ON/OFF states performance and the frequency of their occurrence. In addition, Table IV summarizes the results showing the probability with which the MHSE will result to more accurate control compared to model-based prediction approach. In the worst case, i.e. high noise and the moderate trajectory, the MHSE improves controller performance with a 0.67 probability. In case of low noise and the aggressive trajectory, improvement occurs with a probability as high as 0.99. As expected, increasing process noise deteriorates the performance improvement. Interestingly, the control performance is better with the aggressive trajectory. This observation is in accordance with [6], which also uses an MHSE approach. Last, note that there are cases in which the MHSE method leads to worse controller performance than the prediction approach. In some cases, such as those in Fig. 8c, this is mainly the result of poor TCL state estimates due to high noise and little information retrieval while tracking the moderate trajectory. Also, in small TCL aggregations recent control actions influence a lot the tracking capability in the next time steps, which may result in performance deterioration. This explains, for example, the outliers with negative controller improvement in Fig. 8d and Fig. 8e.
where
\[
E_4 = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}^T,
\]
\[
E_5 = [M_x - m - \epsilon - m_x \ M - \epsilon \ M_u]T,
\]
where \(m_x = 0\) and \(M_x = 100\) are lower and upper bounds on the continuous state \(x_{c,t}\), \(m_u = 1\) and \(M_u = 30\) are lower and upper bounds on \(u_t\), and \(\epsilon = 10^{-6}\) is a small tolerance.

If the power consumption is constant when the TCL is ON, i.e. \(u_t = 1\), the auxiliary variable \(z_t = \delta_{4,t}\) is redundant and (11) can be simplified as
\[
\hat{E}_1 \delta_t \leq \hat{E}_2 x_t + \hat{E}_3,
\]
where \(\hat{E}_2 = E_4, \hat{E}_3 = E_1 + E_5\). The first three columns of \(\hat{E}_1\) are identical to the first three columns of \(E_2\). The forth column of \(\hat{E}_1\) is denoted by \(\hat{E}_1(\cdot, 4)\) and is calculated as
\[
\hat{E}_1(\cdot, 4) = E_2(\cdot, 4) + E_3,
\]
where \(E_2(\cdot, 4)\) denotes the forth column of \(E_2\).

REFERENCES


